Navier-Stokes-like equations for traffic flow

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The macroscopic traffic flow equations derived from the reduced Paveri-Fontana equation are closed starting with the maximization of the informational entropy. The homogeneous steady state taken as a reference is obtained for a specific model of the desired velocity and a kind of Chapman-Enskog method is developed to calculate the traffic pressure at the Navier-Stokes level. Numerical solution of the macroscopic traffic equations is obtained and its characteristics are analyzed.

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I. INTRODUCTION

Traffic flow problems have attracted the attention of many researchers because of the great variety of phenomena present in the motion of vehicles along a highway or in urban networks. The complexity of their behavior constitutes an interesting challenge not only from the practical, but also from a fundamental point of view. Several approaches to study traffic problems can be found in the literature [1–4]. Macroscopic models consider traffic problems as a compressible flow to be described by macroscopic variables such as the density, the average velocity, and the velocity variance. The time evolution of the variables corresponds to balance equations in which some parameters are introduced to take into account the drivers or the highway characteristics. Despite the criticisms against macroscopic models, we must say that they allow for a general understanding of some traffic phenomena.

In the literature there are also some schemes in which a gas-kinetic approach has been developed. Prigogine [5] wrote a kinetic equation for a velocity distribution function taking into account a collective relaxation for it. It is well known that such kinetic equation has some shortcomings, which have been overtaken by the Paveri-Fontana [6] kinetic equation. Though the complete Paveri-Fontana equation has not been solved even for simple cases, it has been studied to support macroscopic approaches like the ones worked by Helbing [7] and Wagner [8]. The Paveri-Fontana equation describes the time evolution of a distribution function in a phase space where both the instantaneous as well as the desired velocities for each vehicle play a role. It considers an individual relaxation of the instantaneous velocity to the desired velocity with a given relaxation time for each vehicle and a binary interaction term.

The macroscopic equations for the relevant variables can be derived from the kinetic equation averaging over the instantaneous velocity. This is a well-known procedure in kinetic theory [9], and its application in traffic flow problems drives us to the variables mentioned before. The method is analogous to the derivation of the Euler and Navier-Stokes equations in fluids theory and we must say that the closure problem is also present. It means that there are some quantities which must be evaluated with constitutive relations in order to obtain a set of closed equations. The analogy with the Chapman-Enskog method in kinetic theory gives us a clue to proceed, provided we have an analogy with the equilibrium, or at least the local equilibrium distribution function.

In this work we propose the use of the maximization of the informational entropy to construct the basic distribution function to be used as the equivalent of a zeroth-order distribution function. The informational entropy to be maximized is restricted by the values of the macroscopic variables we have chosen to describe the system [10]. Accordingly, the distribution function consistent with the maximization of the informational entropy allows for the calculation of the constitutive quantities needed to complete the closure hypothesis. All these calculations can be made provided we introduce a model for the desired velocity, it means that we must make a guess about the average behavior in the desired velocity of drivers.

This procedure gives us a set of equations which we called as Euler equations, because the calculated traffic pressure is given in terms of the density and the velocity but not in terms of spatial gradients. Also an extension of a kind of Chapman-Enskog method and a BGK collective time approximation for the interaction term in the Paveri-Fontana equation have allowed us to construct a corrected distribution function. It contains the collective relaxation time as a parameter but otherwise is given in terms of density, velocity and the gradient of the velocity. The correction for the traffic pressure becomes proportional to the velocity gradient in such a way that we can follow the similarity with fluid theory and call this approximation the Navier-Stokes regime. The macroscopic equations resulting from this procedure are the main goal of this paper, where we derive and solve them for some initial conditions.

The paper is organized as follows: In Sec. II we briefly recall the Paveri-Fontana equation, while Sec. III will be devoted to the construction of the macroscopic traffic equations. In Sec. IV we obtain the distribution function corresponding to a homogenous and steady state as well as the informational entropy for an arbitrary state in the system. In Sec. V we introduce the analogous of the Chapman-Enskog method to calculate a first order correction and close the

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macroscopic set of equations, whereas in Sec. VI we study the linear stability for the resulting macroscopic equations. In Sec. VII we give the numerical solution of the traffic equations for two sets of initial conditions and finally in Sec. VIII we give some remarks.

II. THE PAVERI-FONTANA EQUATION

In order to correct some deficiencies of the gas-kinetic traffic model proposed by Prigogine and co-workers [5], Paveri-Fontana developed a Boltzmann-type treatment for traffic flow which takes into account individual driver's acceleration behavior [6]. In the Paveri-Fontana model the traffic state is characterized by the one-vehicle distribution function $g(x,c,w,t)$ such that $g(x,c,w,t)dxdcdw$ gives at time t the number of vehicles in the road interval between x and $x+dx$ and in the (actual) velocity interval between c and $c+dc$ with desired velocity between w and $w+dw$. For a unidirectional single-lane road on which passing is allowed to occur, the distribution function satisfies the following gas-kinetic traffic equation:

$$\frac{\partial g}{\partial t}+c\frac{\partial g}{\partial x}+\frac{\partial}{\partial c}\left(g\frac{dc}{dt}\right)+\frac{\partial}{\partial w}\left(g\frac{dw}{dt}\right) = f(x,c,t)\int_c^\infty (1-\mathsf{p})(c'-c)g(x,c',w,t)dc' - g(x,c,w,t) \times\int_0^c (1-\mathsf{p})(c-c')f(x,c',t)dc', \tag{1}$$

where

$$f(x,c,t)=\int_0^\infty g(x,c,w,t)dw \tag{2}$$

is the one-vehicle velocity distribution function.

The right-hand side of Eq. (1) is the so-called interaction (or collision) term and describes the deceleration processes which are caused by slower vehicles that cannot be immediately overtaken. The first part of the interaction term corresponds to situations where a vehicle with velocity c' must decelerate to velocity c causing an increase of the distribution function, while the second one describes the decrease of the distribution function due to situations in which vehicles with velocity c must decelerate to even slower velocity c'. In the derivation of the interaction term the following assumptions have been made:

(1) vehicles are regarded as pointlike objects;

(2) a slower vehicle can be immediately overtaken with the probability p;

(3) the velocity of a slow vehicle is not affected by interactions or by the fact of being passed;

(4) the slowing down process is instantaneous, i.e., there is no braking time;

(5) only two-vehicle interactions have been considered; and

(6) the two-vehicle distribution function is factorized in one-vehicle distribution functions, in such a way that a kind of vehicular chaos is assumed.

In contrast to Prigogine, an individual relaxation process was enclosed by Paveri-Fontana in the acceleration term appearing on the left-hand side of the gas-kinetic traffic equation. Assuming that drivers approach their (constant) desired velocity exponentially in time with a constant relaxation time τ, we can write

$$\frac{dc}{dt}=\frac{w-c}{\tau} \quad \text{and} \quad \frac{dw}{dt}=0. \tag{3}$$

The acceleration law $(3)_1$ seems to be a good approximation since most drivers gradually reduce the acceleration as they approach their desired velocity.

The main shortcoming of Paveri-Fontana's traffic equation is the great difficulty encountered in seeking analytical solutions in all cases in which the interaction process cannot be neglected. To overcome this difficulty, we integrate equation (1) with respect to desired velocity and obtain the reduced Paveri-Fontana equation

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}+\frac{\partial}{\partial c}\left(f\frac{V_0-c}{\tau}\right) = f(x,c,t)\int_0^\infty (1-\mathsf{p})(c'-c)f(x,c',t)dc', \tag{4}$$

where

$$V_0(x,c,t)=\int_0^\infty w\frac{g(x,c,w,t)}{f(x,c,t)}dw \tag{5}$$

is the desired average velocity for vehicles moving with the actual velocity c. Besides, in the derivation of the reduced Paveri-Fontana equation (4) we have asked the one-vehicle distribution function $g(x,c,w,t)$ to satisfy the following boundary conditions:

$$\lim_{w\to 0} g(x,c,w,t)=0 \quad \text{and} \lim_{w\to\infty} g(x,c,w,t)=0. \tag{6}$$

Closing this section we want to remark that the main difference between the reduced Paveri-Fontana's traffic equation and the Prigogine's formulation lies in the acceleration term which in the Prigogine equation is modeled by a collective relation term towards an equilibrium distribution function.

III. MACROSCOPIC TRAFFIC EQUATIONS

The reduced Paveri-Fontana traffic equation allows the derivation of dynamic equations for macroscopic quantities like the vehicular density

$$\rho(x,t)=\int_0^\infty f(x,c,t)dc \tag{7}$$

and the average velocity

$$V(x,t)=\int_0^\infty c\frac{f(x,c,t)}{\rho(x,t)}dc. \tag{8}$$

The integration of the reduced Paveri-Fontana traffic equation (4) over all values of the actual velocity c yields the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = 0, \tag{9}$$

while the velocity equation

$$\rho\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}\right) + \frac{\partial \mathcal{P}}{\partial x} = \rho\frac{W-V}{\tau} - \rho(1-\mathsf{p})\mathcal{P} \tag{10}$$

can be obtained if we multiply the reduced Paveri-Fontana equation with c and integrate over all values of the actual velocity. In the above equation we have introduced the average desired velocity

$$W(x,t) = \int_0^\infty V_0(x,c,t)\frac{f(x,c,t)}{\rho(x,t)}dc \tag{11}$$

and the so-called traffic pressure

$$\mathcal{P}(x,t) = \int_0^\infty (c-V)^2 f(x,c,t)dc, \tag{12}$$

which is related to the velocity variance $\Theta(x,t)$ through the relation

$$\mathcal{P}(x,t) = \rho(x,t)\Theta(x,t). \tag{13}$$

At this point it is important to emphasize that the macroscopic traffic equations (9) and (10) can only be obtained if the one-vehicle velocity distribution function $f(x,c,t)$ satisfies the boundary conditions

$$\lim_{c\to 0} f(x,c,t) = 0 \quad \text{and} \lim_{c\to\infty} f(x,c,t) = 0. \tag{14}$$

In order to derive a macroscopic traffic model analogous to the Navier-Stokes description for ordinary fluids, we shall assume that all macroscopic information of the system can be obtained from the vehicular density and the average velocity, i.e., we construct a traffic model based only on these variables. However, the traffic pressure is not known in terms of them, it means that the set of equations we have just derived is not closed, a fact which prevents us to find a solution. To obtain a closed set of equations we need a constitutive relation for the traffic pressure in terms of the vehicular density, average velocity and their corresponding spatial gradients. The similarity of this problem with the description of fluids invites us to name this set of equations as Euler-type traffic equations when we write the constitutive equations in terms of the vehicular density and average velocity, and Navier-Stokes-like traffic equations when they are written also in terms of the first-order spatial gradients.

For this purpose, we shall apply an approximation similar to the Chapman-Enskog method and find a first-order approximation for the distribution function which solves the reduced Paveri-Fontana traffic equation. In the Chapman-Enskog method we write the velocity distribution function as

$$f(x,c,t) = f^{(0)}(x,c,t) + f^{(1)}(x,c,t) + \dots , \tag{15}$$

where $f^{(i)}(x,c,t)$ represents sucessive approximations for the distribution function. Assuming that the values of the vehicular density and average velocity are determined only by the zeroth-order approximation for the distribution function, we can easily derive the following conditions for $i \geqslant 1$:

$$\int_0^\infty f^{(i)}(x,c,t)dc = 0 \quad \text{and} \int_0^\infty cf^{(i)}(x,c,t)dc = 0. \tag{16}$$

However to go further we need the zeroth-order distribution function $f^{(0)}(x,c,t)$, which in this case will be determined according to a maximization procedure for the informational entropy.

IV. THE INFORMATIONAL ENTROPY

First of all, let us consider the homogeneous steady state of the system as a reference to construct the informational entropy. The distribution function corresponding to this homogeneous steady state will be the solution of the reduced Paveri-Fontana equation when there is no dependence on time and space,

$$\frac{\partial}{\partial c}\left(f_e(c)\frac{V_0(c)-c}{\tau}\right) = \rho_e(1-\mathsf{p})(V_e - c)f_e(c), \tag{17}$$

where $f_e(c)$ is the distribution function we are looking for. Moreover, the vehicular density ρ_e and the average velocity V_e corresponding to this homogeneous steady state are given by

$$\rho_e = \int_0^\infty f_e(c)dc \quad \text{and} \quad \rho_e V_e = \int_0^\infty c f_e(c)dc. \tag{18}$$

To find the stationary and homogeneous solution of the reduced Paveri-Fontana equation (17) we must provide an expression for the desired average velocity $V_0(c)$ of vehicles moving with actual velocity c. Here, we shall assume that

$$V_0(c) = \omega c \quad (\omega > 1), \tag{19}$$

where ω is a positive constant. Relation (19) indicates that drivers desired velocity increases as their actual velocity increases, i.e., drivers want to drive even more and more fast. It should be noticed that relation (19) represents a model for an average over the desired velocity of drivers as can be seen in Eq. (5). It is clear that this is not a unique model but just a sound one. Though this model may produce desired velocities tending to infinity, let us recall that the distribution function goes to zero as the velocity increases so that the number of vehicles with velocities tending to infinity goes to zero. Hence, the solution of Eq. (17) leads to the following expression for the homogeneous steady distribution function:

$$f_e(c) = \frac{\alpha}{\Gamma(\alpha)}\frac{\rho_e}{V_e}\left(\frac{\alpha c}{V_e}\right)^{\alpha-1}\exp\left(-\frac{\alpha c}{V_e}\right), \tag{20}$$

where

$$\alpha = \frac{\rho_e(1-\mathsf{p})V_e\tau}{\omega - 1}, \tag{21}$$

is a dimensionless constant characteristic of the homogeneous steady state and $\Gamma(\alpha)$ is the gamma function. Note that the constant α depends on several traffic parameters like the relaxation time, the probability of overtaking, the model constant we are using for the average desired velocity, the vehicular density and the average velocity of the homogeneous steady state.

Now let us define the informational entropy of this system in terms of the zeroth-order distribution function, relative to the homogeneous steady state, as

$$s(x,t) = -\int_0^\infty f^{(0)}(x,c,t)\ln\left(\frac{f^{(0)}(x,c,t)}{f_e(c)}\right)dc. \qquad (22)$$

We notice that this informational entropy is in fact a change in entropy with respect to the entropy in the steady state [11–13]. The zeroth-order approximation for the one-vehicle velocity distribution function follows through a maximization procedure of the informational entropy by taking into account the restrictions imposed by the values of the macroscopic variables that we have chosen to describe the system. In the traffic model we present here we shall take the density and the average velocity of the system to be known. The distribution function must be consistent with the values of these variables, which in fact are functions of road position and time. Those variables are known in principle from the measurements taken for special case studies, though in this work we determine them by solving the set of resulting equations. Hence, we construct the following functional of the distribution function:

$$\mathcal{F} = -\int_0^\infty (\ln f^{(0)}(x,c,t) - \ln f_e(c) + \beta + \lambda c) f^{(0)}(x,c,t)dc, \qquad (23)$$

where the Langrange multipliers β and λ depend on position and time. The maximization procedure allows us to find a distribution function which satisfies the condition $\delta\mathcal{F}/\delta f^{(0)} = 0$ and it is given by

$$f^{(0)}(x,c,t) = f_e(c)\exp(-1-\beta-\lambda c). \qquad (24)$$

The determination of the Lagrange multipliers follows from restrictions (7) and (8) which we have imposed in the maximization procedure of the informational entropy. However, we verify that such determination requires the knowledge of the homogeneous steady distribution function. As we have seen above, the homogeneous steady state distribution function depends on the model chosen for the average desired velocity. This means that all properties we can deduce from this scheme must be consistent with the model. Taking into account the expression (20) for the homogeneous steady distribution function, the direct substitution of the velocity distribution function (24) into the definitions of the vehicular density and average velocity drives us to

$$\lambda = \frac{\alpha}{V}\left(1 - \frac{V}{V_e}\right) \quad \text{and} \quad \exp(1+\beta) = \frac{\rho_e}{\rho}\left(\frac{V}{V_e}\right)^{\alpha}. \qquad (25)$$

Hence, the zeroth-order velocity distribution function becomes

$$f^{(0)}(x,c,t) = \frac{\alpha}{\Gamma(\alpha)}\frac{\rho}{V}\left(\frac{\alpha c}{V}\right)^{\alpha-1}\exp\left(-\frac{\alpha c}{V}\right), \qquad (26)$$

where we can notice that the structure of this zeroth-order distribution function is the same as the one for the homogeneous steady state. In fact it resembles what we immediately identify with a kind of local distribution function, since the velocity distribution function (26) can be obtained from the homogeneous distribution (20) by replacing the density ρ_e by $\rho(x,t)$ and the velocity V_e by $V(x,t)$, i.e., by their local values along the highway. However it is important to emphasize that expression (26) comes from the maximization of the informational entropy.

Consistently with this zeroth-order approximation for the velocity distribution function we calculate the zeroth-order relation for the traffic pressure, i.e.,

$$\mathcal{P} = \frac{\rho V^2}{\alpha}. \qquad (27)$$

From Eqs. (13) and (27) we can see that the velocity variance $\Theta = V^2/\alpha$ in the zeroth-order approximation is proportional to the square of the average velocity, a fact which can be observed in empirical traffic data [3,14].

Insertion of the constitutive relation (27) for the traffic pressure into Eqs. (9) and (10) leads to the so-called Euler-type traffic equations

$$\frac{d\rho}{dt} = -\rho\frac{\partial V}{\partial x}, \qquad (28)$$

$$\frac{dV}{dt} = -\frac{V^2}{\alpha\rho}\frac{\partial\rho}{\partial x} - \frac{2V}{\alpha}\frac{\partial V}{\partial x} + \frac{\omega-1}{\tau}V - \rho(1-\mathsf{p})\frac{V^2}{\alpha}, \qquad (29)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x} \qquad (30)$$

denotes the material time derivative. The set of Eqs. (28) and (29) are now closed and in principle can be solved for a given initial condition. This set is called the Euler-type traffic equation because the traffic pressure is given only in terms of the density and the average velocity, which are the macroscopic variables chosen to describe the problem. It resembles the hydrostatic pressure in fluids where the local equation of state is given in terms of local variables only. A linear stability analysis of the set of equations showed that the homogeneous steady state is unstable under small perturbations. Then to go further we must calculate at least a first-order distribution function which takes us away from the local state described by the zeroth-order distribution function.

V. NAVIER-STOKES-LIKE TRAFFIC EQUATIONS

In order to determine now the first-order approximation for the velocity distribution function we introduce the expansion (15) into the reduced Paveri-Fontana equation. Utilizing the fact that $f^{(1)}(x,c,t)$ will usually be small compared to $f^{(0)}(x,c,t)$ we get

$$f^{(0)}\left[\alpha\left(\frac{c}{V}-1\right)^2-2\left(\frac{c}{V}-1\right)-1\right]\frac{\partial V}{\partial x}$$
$$=-\int_0^\infty L(c,w|x,t)f^{(1)}(x,w,t)dw, \quad (31)$$

where

$$L(c,w|x,t)=\rho(1-\mathsf{p})(c-V)\delta(c-w)$$
$$+(1-\mathsf{p})f^{(0)}(x,c,t)(c-w). \quad (32)$$

In the derivation of the above expression we have neglected all nonlinear terms and eliminated the material time derivatives by using the Euler-type traffic equations. If we apply the so-called relaxation time approximation (see Helbing [7])

$$\int_0^\infty L(c,w|x,t)f^{(1)}(x,w,t)dw \approx \frac{f^{(1)}}{\tau_0}, \quad (33)$$

we get the following expression for the first-order approximation of the distribution function:

$$f^{(1)}(x,c,t)=-f^{(0)}\tau_0\left[\alpha\left(\frac{c}{V}-1\right)^2-2\left(\frac{c}{V}-1\right)-1\right]\frac{\partial V}{\partial x}, \quad (34)$$

where τ_0 is the mean free interaction time, i.e., the average time between sucessive vehicular interactions.

With Eqs. (15), (26), and (34) we can now calculate the first-order relation for the traffic pressure, namely,

$$\mathcal{P}=\frac{\rho V^2}{\alpha}\left(1-\tau_*\frac{\partial V}{\partial x}\right), \quad (35)$$

where

$$\tau_* = 2\tau_0\frac{1+\alpha}{\alpha} \quad (36)$$

plays the role of an effective relaxation time. It measures the relevance of the deviation of the traffic pressure form the Euler value which can be seen as the equivalent of the hydrostatic pressure. Note that the above constitutive relation for the vehicular traffic pressure has a similar form to the Navier-Stokes relation for ordinary fluids since in non-equilibrium situations both depend on the velocity gradient. In fact, a viscosity coefficient can be identified as $\eta=\rho V^2\tau_*/\alpha$ which is a function of the state of the system through the density and the average velocity.

The Navier-Stokes-like set of equations are given by the density equation (28) and the velocity equation, which now is written as follows:

$$\frac{dV}{dt}=-\frac{V^2}{\alpha\rho}\left(1-\tau_*\frac{\partial V}{\partial x}\right)\left(\frac{\partial\rho}{\partial x}+2\frac{\rho}{V}\frac{\partial V}{\partial x}+\rho^2(1-\mathsf{p})\right)$$
$$+\frac{\tau_*}{\alpha}V^2\frac{\partial^2 V}{\partial x^2}+\frac{\omega-1}{\tau}V. \quad (37)$$

We remark that τ_* depends on the collective relaxation time τ_0 we introduced to calculate the collision term in the kinetic equation. Hence, our model contains two free adjustable parameters, ω for the desired velocity model and τ_0 for the collision term.

VI. LINEAR STABILITY ANALYSIS

Our aim in this section is to determine if the Navier-Stokes-like traffic equations predicts unstable traffic (e.g., the formation of density clusters and stop-and-go traffic) in the range of densities where the Paveri-Fontana equation and consequently our model is valid. The instability region is found via a linear stability analysis with [15]

$$\frac{\rho(x,t)-\rho_e}{\rho_e}=\bar{\rho}\exp(i\kappa x+\gamma t) \quad \text{and} \quad (38)$$

$$\frac{V(x,t)-V_e}{V_e}=\bar{V}\exp(i\kappa x+\gamma t),$$

where κ is the wave number of the perturbations and γ is the growth parameter.

Inserting the above small perturbations into the Navier-Stokes-like equations (28) and (37), applying Taylor expansion, and neglecting nonlinear contributions we get the following system of algebraic equations:

$$\begin{pmatrix}\gamma+i\beta & i\beta\\ i\dfrac{\beta}{\alpha}+2(\omega-1) & \gamma+i\beta\left(\dfrac{2+\alpha}{\alpha}\right)+\dfrac{\tau_*\beta^2}{\alpha}+(\omega-1)(1-i\beta\tau_*)\end{pmatrix}$$
$$\times\begin{pmatrix}\bar{\rho}\\ \bar{V}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \quad (39)$$

Here, the time is measured in units of τ, the length in units of $1/\hat{\rho}$, the velocity in units of $1/\tau\hat{\rho}$ and the vehicle density in units of $\hat{\rho}$, where $\hat{\rho}$ is the maximum vehicular density. Moreover, we have introduced the abbreviation $\beta=\kappa V_e$ and assumed that the probability of passing takes the explicit form [5]

$$\mathsf{p}=1-\frac{\rho}{\hat{\rho}}. \quad (40)$$

The system of algebraic equations (39) has a nontrivial solution if the determinant of the coefficients vanishes. This condition leads to the dispersion relation

$$\gamma^2+B(\kappa)\gamma-C(\kappa)=0, \quad (41)$$

where

$$B(\kappa)=2i\beta\left(\frac{1+\alpha}{\alpha}\right)+\frac{\tau_*\beta^2}{\alpha}+(\omega-1)(1-i\beta\tau_*) \quad (42)$$

and

$$C(\kappa)=\beta^2\left[\left(\frac{1+\alpha}{\alpha}\right)-(\omega-1)\tau_*\right]+i\beta\left[(\omega-1)-\frac{\tau_*\beta^2}{\alpha}\right]. \quad (43)$$

The dispersion relation (41) is satisfied for two complex roots which can be determined analytically, i.e.,

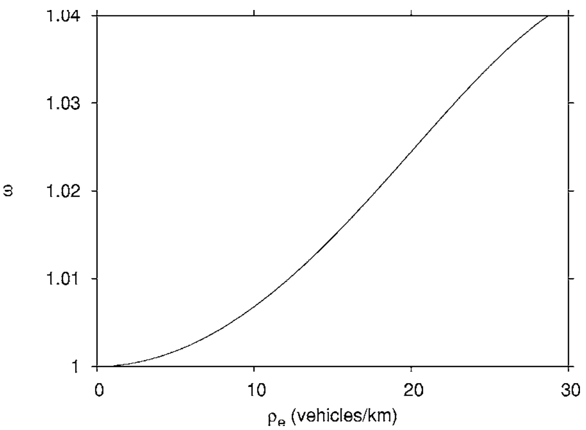

FIG. 1. The parameter ω as a function of the vehicular density.

$$\gamma_{\pm}(\kappa) = -\frac{B}{2} \pm \sqrt{\left(\frac{B}{2}\right)^2 + C}. \tag{44}$$

The Navier-Stokes-like traffic equations predicts an unstable solution if the real part of the growth parameter of at least one of the roots of the dispersion relation has a positive real part, i.e., if

$$\mathrm{Re}(\gamma_+) > 0 \quad \text{or } \mathrm{Re}(\gamma_-) > 0. \tag{45}$$

In fact, we verify that $\mathrm{Re}(\gamma_-) < 0$ for all values of the free parameters β and ω in the dispersion relation. However the root γ_+ shows some regions in which its real part becomes positive, hence we have some instability regions. A note must be said for the calculation of the range of values for the parameter ω, which is the one introduced in the model for the average desired velocity of drivers. First we notice that the experimental data for the variance of the velocity show a quadratic dependence with the average velocity, and for low density the prefactor is almost constant [7,14]. This fact agrees with our calculation in the sense that the velocity variance we found is quadratic in the average velocity and the prefactor is the dimensionless constant we call α. Hence, the quantity α can be taken from the experimental data and in this work we will use $\alpha = 100$. We can also infer from relation (40) that the factor $(1-\mathsf{p})$ in a homogeneous steady state is only a function of the homogeneous density. On the other hand, the empirical velocity-density relations allow us to write the average velocity as $V_e = V_e(\rho_e) = V_{\max}(\{1 + \exp[(\rho/\hat{\rho} - 0.25)/0.06]\}^{-1} - 3.72 \times 10^{-6})$ in such a way that the dimensionless constant α is written only in terms of ρ_e. Hence, from Eq. (21) we can write $\omega(\rho_e) = 1 + \rho_e(1 - \mathsf{p})\tau V_e(\rho_e)/\alpha$ which allow us the calculation of the quantity ω we are looking for. In this way we can find the range of values for ω in the low density region, Fig. 1 shows its resulting behavior. We notice that $\omega > 1$ as it should be according to the requirements of our model, however though the range of values is very short we must recall that it represents an average and in this sense it can be seen as a natural feature of this model. Also, it should be remembered that the number of vehicles with a very big velocity tends to zero consistently with the behavior of the distribution function.

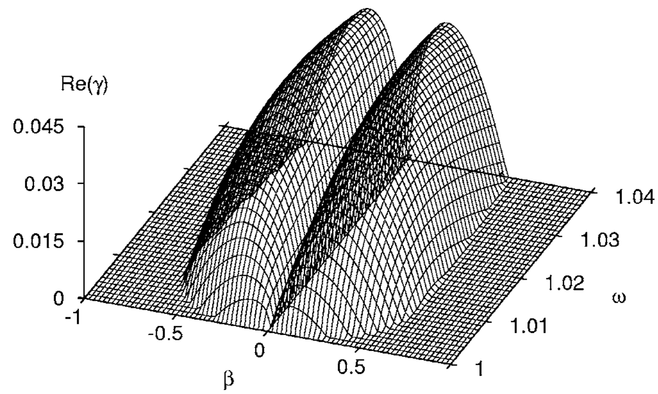

FIG. 2. Instability regions for $\tau_0/\tau = 10$.

Figures 2 and 3 illustrate the instability regions in the (β, ω) plane—given that the road has an infinite length—for the values $\tau_0 = 10\tau$ and $\tau_0 = 20\tau$, respectively. We notice that the instability region shrinks as the value of the collective relaxation time τ_0 grows. It means that τ_0 is a stabilization factor and in the constitutive relation (35) we see that it is also a measure of the viscosity in the system. Its origin being the interaction term in the Paveri-Fontana equation and the kind of BGK approximation we have made.

VII. NUMERICAL SIMULATION

In order to investigate the numerical solution of the Navier-Stokes-like traffic equations we write (28) and (37) in the following conservative form:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = s(u), \tag{46}$$

where

$$u = \begin{pmatrix} \rho \\ \rho V \end{pmatrix}, \quad F(u) = \begin{pmatrix} \rho V \\ \rho V^2 + \mathcal{P} \end{pmatrix}, \quad \text{and}$$

$$s(u) = \begin{pmatrix} 0 \\ \rho \dfrac{\omega - 1}{\tau} V - \rho(1 - \mathsf{p})\mathcal{P} \end{pmatrix}. \tag{47}$$

For the explicit numerical simulation of the macroscopic traffic equations (46) we discretize position and time on a

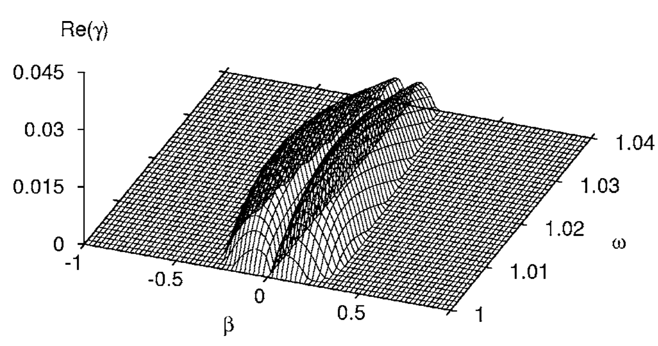

FIG. 3. Instability regions for $\tau_0/\tau = 20$.

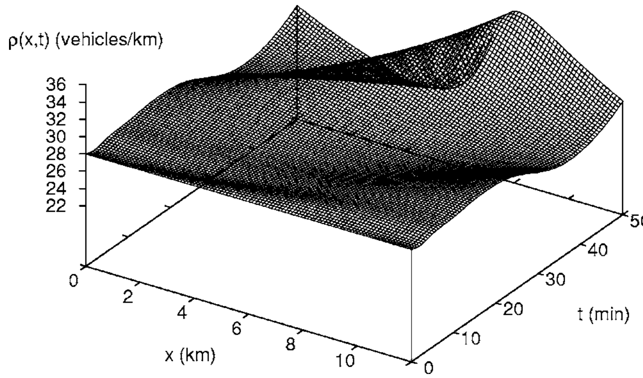

FIG. 4. Spatiotemporal behavior of the vehicular density.

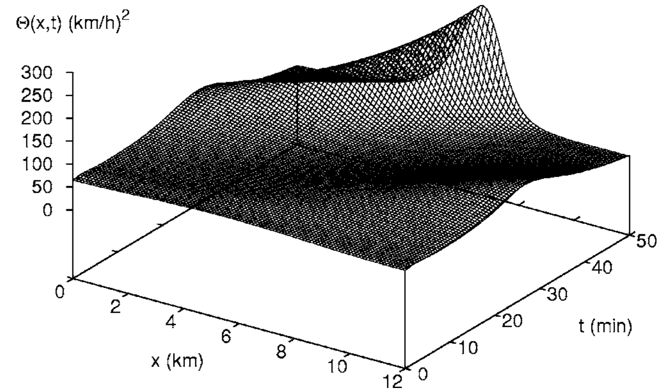

FIG. 6. Spatiotemporal behavior of the velocity variance.

uniform grid with values $x_i=i\Delta x(i=0,1,2,\ldots)$ and $t_n=n\Delta t(n=0,1,2,\ldots)$, respectively. Hence, we can determine $u_i^n=u(x_i,t_n)$ at the grid points by using finite difference methods [16–18]. Here, we apply the two-step Lax-Wendroff method,

$$u_{i+1/2}^{n+1/2}=\frac{1}{2}\left(u_i^n+u_{i+1}^n-\frac{\Delta t}{\Delta x}(F_{i+1}^n-F_i^n)+\frac{\Delta t}{2}(s_i^n+s_{i+1}^n)\right)\quad\text{(predictor)},\tag{48}$$

$$u_i^{n+1}=u_i^n-\frac{\Delta t}{\Delta x}(F_{i+1/2}^{n+1/2}-F_{i-1/2}^{n+1/2})+\frac{\Delta t}{2}(s_{i+1/2}^{n+1/2}+s_{i-1/2}^{n+1/2})\quad\text{(corrector)}.\tag{49}$$

Numerical solutions of macroscopic traffic models (partial differential equations) require the specification of the initial and boundary conditions. For reasons of simplicity we have used periodic boundary conditions, $\rho(0,t)=\rho(L,t)$ and $V(0,t)=V(L,t)$, where L is the length of the road. As initial conditions we have considered two cases.

Case 1: Let us consider a steady and homogeneous traffic state plus a small perturbation on the average velocity which reflects the fact that some vehicles drive a little faster, while others move a little slower than the homogeneous steady velocity. Accordingly the initial conditions for the density and the velocity can be written as [7]

$$\rho(x,0)=\rho_e\quad\text{and } V(x,0)=V_e(\rho_e)+\delta V\sin\left(\frac{2\pi x}{L}\right).\tag{50}$$

The values of the model parameters taken for the implementation of the numerical method are $\rho_e=28$ vehicles/km, $V_e(\rho_e)=84$ km/h, $\hat{\rho}=140$ vehicles/km, $\delta V=0.84$ km/h, $\tau=30$ s, $\tau_0=300$ s, and $L=12$ km. The obtained solution for the density is shown in Fig. 4, whereas in Figs. 5 and 6 we show the average velocity and the corresponding velocity variance, respectively.

Case 2: We assume now a homogeneous steady traffic state and add to the density a localized perturbation of amplitude $\delta\rho$ so that the initial conditions are given by [17]

$$\rho(x,0)=\rho_e+\delta\rho\left[\cosh^{-2}\left(\frac{x-x_0}{\omega_+}\right)-\frac{\omega_+}{\omega_-}\cosh^{-2}\left(\frac{x-x_0-\Delta x_0}{\omega_-}\right)\right]\tag{51}$$

and

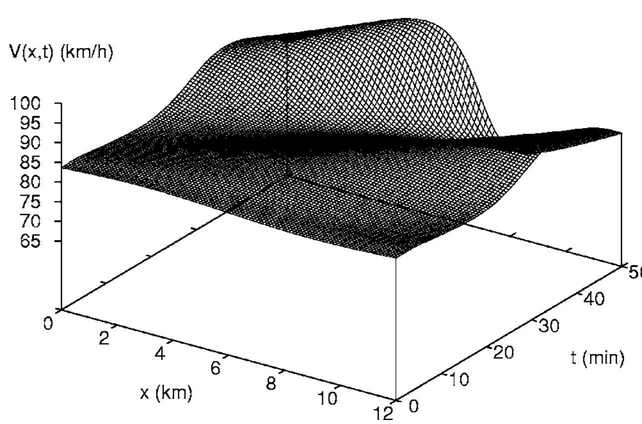

FIG. 5. Spatiotemporal behavior of the average velocity.

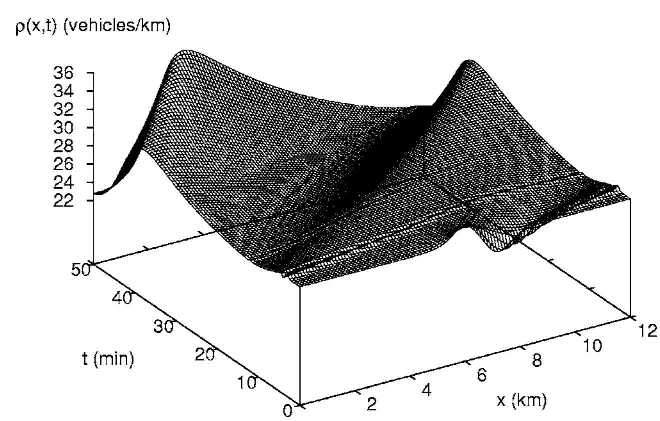

FIG. 7. Spatiotemporal behavior of the vehicular density.

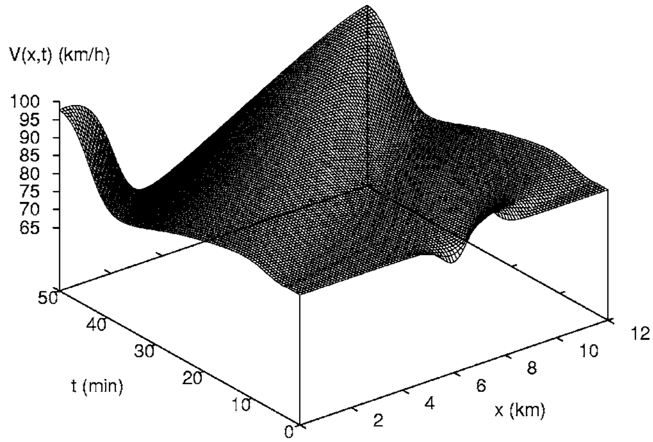

FIG. 8. Spatiotemporal behavior of the average velocity.

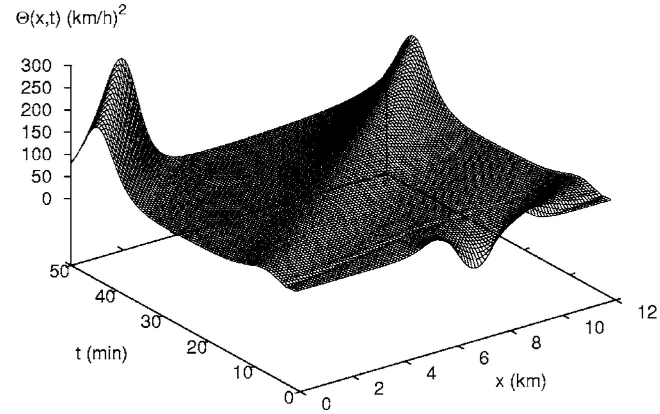

FIG. 9. Spatiotemporal behavior of the velocity variance.

$$\rho(x,0)V(x,0) = \rho_e V_e(\rho_e), \tag{52}$$

where x_0 and $x_0+\Delta x_0$ denote the positions of the positive and negative peaks of the perturbation with widths ω_+ and ω_-, respectively. Here, we have taken the values for the model parameters given before and $\delta\rho=2$ vehicles/km, $x_0=6$ km, $\omega_+=\omega_-=500$ m, and $\Delta x_0=\omega_++\omega_-=1000$ m. Figures 7–9 show the spatiotemporal behavior of the density, average velocity, and velocity variance corresponding to this second case.

In both cases we observe that an increase in the density at a certain road position is related to a reduction of the average velocity at the same point. We notice that the maximum density obtained corresponds to the region of low to moderate densities, it does not go beyond 36 vehicles/km. In the first case and for short times the changes in density are very smooth but the velocity variance is sensible to the velocity perturbation. In the second case both variables follow each other. On the other hand, the maximum density is obtained for distances larger than the maximum in the variance, it means that a change in the variance precedes a maximum in the density along the road. Also the peaks in the variance are sharper than the corresponding ones in the density.

VIII. CONCLUDING REMARKS

The model presented in this work starts with the mesoscopic approach for traffic flow problems based on the reduced Paveri-Fontana kinetic equation. The homogeneous steady solution for a particular model in the average desired velocity for the drivers is obtained and it constitutes the reference state to study the vehicular flow. Though the calculations are made for a specific model, it is realistic enough since the drivers natural tendency corresponds to drive at a faster speed than its actual one. In a kind of Chapman-Enskog method to solve the reduced Paveri-Fontana equation the zeroth order reference state is calculated by means of the maximization of the informational entropy restricted by the values of the density and the average velocity in the system. To obtain the first-order distribution function we used the reduced Paveri-Fontana equation with a relaxation time approximation for the interaction term. This procedure allowed us to find a traffic pressure with a kind of Euler term in which we have a proportionality of the variance with the square of the average velocity, a fact which is consistent with the experimental data. Also, in the first order there appears a term proportional to the spatial derivative of the velocity, it is interpreted as a Navier-Stokes viscosity term. The viscosity is not a constant but depends on the state of the system through the density, the average velocity and contains the collective relaxation time introduced in the approximation method.

The macroscopic equations for the density and the average velocity are classified as Navier-Stokes-like traffic equations. They have some instability regions obtained from a linear stability analysis and the solutions for the density and the average velocity show the qualitative characteristics obtained for traffic models in the low density region. What we mean is that our model predicts increases in the density followed by a reduction in velocity. As we noticed before the velocity variance precedes the growing in the density along the road. However it is important to notice that the density is always smaller than the maximum density, the average velocity is always positive and the variance value is consistent with the experimental data. Besides in this model we have only the density and average velocity as independent variables, so in this sense it is the simplest model we can construct in terms of two macroscopic variables.

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[1] D. Chowdhury, L. Santen, and A. Schadsneider, Phys. Rep. **329**, 199 (2000).
[2] T. Nagatani, Rep. Prog. Phys. **65**, 1331 (2002).
[3] D. Helbing, Rev. Mod. Phys. **73**, 1067 (2001).
[4] B. S. Kerner, *The Physics of Traffic* (Springer-Verlag, Heidelberg, 2004).
[5] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (American Elsevier, New York, 1971).
[6] S. L. Paveri-Fontana, Transp. Res. **9**, 225 (1975).
[7] D. Helbing, Phys. Rev. E **53**, 2366 (1996); D. Helbing, Physica A **219**, 375 (1995); **219**, 391 (1995); cond-mat/9805136 (unpublished).
[8] C. Wagner, C. Hoffmann, R. Sollacher, J. Wagenhuber, and B. Schürmann, Phys. Rev. E **54**, 5073 (1996).
[9] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1970).
[10] R. M. Velasco and A. R. Méndez, in *Statistical Physics and Beyond*, edited by F. J. Uribe, L. S. García Colín, and E. Díaz Herrera, AIP Conference Proceedings 757 (AIP, New York, 2005).
[11] E. W. Montroll, Proc. Natl. Acad. Sci. U.S.A. **78**, 7839 (1981).
[12] H. Reiss, A. D. Hammerich, and E. W. Montroll, J. Stat. Phys. **42**, 647 (1986).
[13] K. G. Denbigh and J. S. Denbigh, *Entropy in Relation to Incomplete Knowledge* (Cambridge University Press, Cambridge, 1985).
[14] V. Shvetsov and D. Helbing, Phys. Rev. E **59**, 6328 (1999).
[15] B. S. Kerner and P. Konhäuser, Phys. Rev. E **48**, R2335 (1993).
[16] D. Helbing, Phys. Rev. E **51**, 3164 (1995).
[17] D. Helbing and M. Treiber, Comput. Sci. Eng. **1**, 89 (1999).
[18] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, Cambridge, 1992).